

On the lower bound of the discrepancy of Halton's sequence II

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Abstract

Let $(H_s(n))_{n \geq 1}$ be an s -dimensional generalized Halton's sequence. Let D_N^* be the discrepancy of the sequence $(H_s(n))_{n=1}^N$. It is known that $D_N^* = O(\ln^s N)$ as $N \rightarrow \infty$. In this paper, we prove that this estimate is exact. Namely, there exists a constant $C(H_s) > 0$, such that

$$\max_{1 \leq M \leq N} MD_M^* \geq C(H_s) \log_2^s N \quad \text{for } N = 2, 3, \dots$$

Key words: Halton's sequence, ergodic adding machine.

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1 Introduction

Let $(\beta_n)_{n \geq 1}$ be a sequence in the unit cube $[0, 1]^s$, $B_{\mathbf{y}} = [0, y_1] \times \cdots \times [0, y_s]$,

$$\Delta(B_{\mathbf{y}}, (\beta_n)_{n=1}^N) = \sum_{n=1}^N (\mathbb{1}_{B_{\mathbf{y}}}(\beta_n) - y_1 \cdots y_s), \quad (1.1)$$

where $\mathbb{1}_B(\mathbf{x}) = 1$, if $\mathbf{x} \in B$, and $\mathbb{1}_B(\mathbf{x}) = 0$, if $\mathbf{x} \notin B$.

We define the star *discrepancy* of an N -point set $(\beta_n)_{n=1}^N$ as

$$D^*((\beta_n)_{n=1}^N) = \sup_{0 < y_1, \dots, y_s \leq 1} |\Delta(B_{\mathbf{y}}, (\beta_n)_{n=1}^N)/N|. \quad (1.2)$$

In 1954, Roth proved that

$$\limsup_{N \rightarrow \infty} N(\ln N)^{-\frac{s}{2}} D^*((\beta_n)_{n=1}^N) > 0.$$

According to the well-known conjecture (see, e.g., [BeCh, p.283]), this estimate can be improved to

$$\limsup_{N \rightarrow \infty} N(\ln N)^{-s} D^*((\beta_n)_{n=1}^N) > 0. \quad (1.3)$$

In 1972, W. Schmidt proved this conjecture for $s = 1$. For $s = 2$, Faure and Chaix [FaCh] proved (1.3) for a class of (t, s) -sequences. See [Bi] for the most important results on this conjecture.

Definition. An s -dimensional sequence $((\beta_n)_{n \geq 1})$ is of **low discrepancy** (abbreviated l.d.s.) if $D^*((\beta_n)_{n=1}^N) = O(N^{-1}(\ln N)^s)$ for $N \rightarrow \infty$.

Let $p \geq 2$ be an integer

$$n = \sum_{j \geq 1} e_{p,j}(n) p^{j-1}, \quad e_{p,j}(n) \in \{0, 1, \dots, p-1\}, \quad \text{and} \quad \phi_p(n) = \sum_{j \geq 1} e_{p,j}(n) p^{-j}.$$

Van der Corput proved that $(\phi_p(n))_{n \geq 0}$ is a 1-dimensional l.d.s. (see [VC]). Let

$$\hat{H}_s(n) = (\phi_{\hat{p}_1}(n), \dots, \phi_{\hat{p}_s}(n)), \quad n = 0, 1, 2, \dots,$$

where $\hat{p}_1, \dots, \hat{p}_s \geq 2$ are pairwise coprime integers. Halton proved that $(\hat{H}_s(n))_{n \geq 0}$ is an s -dimensional l.d.s. (see [Ha]). For other examples of l.d.s. see e.g. in [BeCh], [FKP], [Ni]. In [Le2] we proved that Halton's sequence satisfies (1.3). In this paper we generalize this result.

Let $Q = (q_1, q_2, \dots)$ and $Q_j = q_1 q_2 \dots q_j$, where $q_j \geq 2$ ($j = 1, 2, \dots$) is a sequence of integers. Consider Cantor's expansion of $x \in [0, 1)$:

$$x = \sum_{j=1}^{\infty} x_j / Q_j, \quad x_j \in \{0, 1, \dots, q_j - 1\}, \quad x_j \neq q_j - 1 \text{ for infinitely many } j.$$

The Q -adic representation of x is then unique. We define the odometer transform

$$T_Q(x) := (x_k + 1)/Q_k + \sum_{j \geq k+1} x_j / Q_j, \quad T_Q^n(x) = T_Q(T_Q^{n-1}(x)), \quad (1.4)$$

$$n = 2, 3, \dots, T_Q^0(x) = x, \text{ where } k = \min\{j \mid x_j \neq q_j - 1\}.$$

For $Q = (q, q, \dots)$, we obtain von Neumann-Kakutani's q -adic adding machine (see, e.g., [FKP]). As is known, the sequence $(T_Q^n(x))_{n \geq 1}$ coincides for $x = 0$ with the van der Corput sequence (see e.g., [FKP, §2.5]).

Let $h_i \geq 1$, $q_{i,j} \geq 2$ be integers ($1 \leq j \leq h_i, 1 \leq i \leq s$), $p_{i,j} \in \{q_{i,1}, \dots, q_{i,h_i}\}$, $g.c.d.(q_{i,k}, q_{j,l}) = 1$ for $i \neq j$, $\mathcal{P}_i = (p_{i,1}, p_{i,2}, \dots)$, $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_s)$,

$$\tilde{P}_{i,0} = 1, \quad \tilde{P}_{i,j} = \prod_{1 \leq k \leq j} p_{i,k}, \quad i \in [1, s], \quad j \geq 1, \quad T_{\mathcal{P}}(\mathbf{x}) = (T_{\mathcal{P}_1}(x_1), \dots, T_{\mathcal{P}_s}(x_s)),$$

$$n = \sum_{j \geq 1} e_{p_{i,j},j}(n) \tilde{P}_{i,j-1}, \quad e_{p_{i,j},j}(n) \in \{0, 1, \dots, p_{i,j} - 1\}, \quad n = 0, 1, \dots, \quad (1.5)$$

$$\varphi_{\mathcal{P}_i}(n) = \sum_{j \geq 1} e_{p_{i,j},j}(n) \tilde{P}_{i,j}^{-1}, \quad H_{\mathcal{P}}(n) = (\varphi_{\mathcal{P}_1}(n), \dots, \varphi_{\mathcal{P}_s}(n)). \quad (1.6)$$

We note that $H_{\mathcal{P}}(n) = T_{\mathcal{P}}^n(\mathbf{0})$ for $n = 0, 1, \dots$.

Let $\Sigma_i = (\sigma_{i,j})_{j \geq 1}$ be a sequence of corresponding permutations $\sigma_{i,j}$ of $\{0, 1, \dots, p_{i,j} - 1\}$ for $j \geq 1$, $\Sigma = (\Sigma_1, \dots, \Sigma_s)$, $\mathbf{x} = (x_1, \dots, x_s)$,

$$\tilde{\Sigma}(\mathbf{x}) = (\tilde{\Sigma}_1(x_1), \dots, \tilde{\Sigma}_s(x_s)), \quad \tilde{\Sigma}_i(x_i) = \sum_{j \geq 1} \sigma_{i,j}(x_{i,j}) / \tilde{P}_{i,j}, \quad x_i = \sum_{j \geq 1} x_{i,j} / \tilde{P}_{i,j}.$$

We consider the following generalization of the Halton sequence (see [Fa],[He],[FKP]):

$$H_{\mathcal{P}}^{\Sigma}(n, \mathbf{x}) = \tilde{\Sigma}(T_{\mathcal{P}}^n(\mathbf{x})), \quad n = 0, 1, 2, \dots \quad (1.7)$$

We note that $(H_{\mathcal{P}}^{\Sigma}(n, \mathbf{x}))_{n \geq 0}$ coincides for $\mathbf{x} = \mathbf{0}$ and $s = 1$ with the Faure sequence S_Q^{Σ} [Fa]. Similarly to [Ni, p.29-31], we get that $(H_{\mathcal{P}}^{\Sigma}(n, \mathbf{x}))_{n \geq 0}$ is of low discrepancy.

2 The Theorem and its proof

In this section we will prove

Theorem. *Let $s \geq 2$, $h_0 = \max_i h_i$, $q_0 = \max_{i,j} q_{i,j}$, $C_1 = 2sh_0q_0^s \log_2 q_0$ and $C = 2^{s+3}s^s h_0^s q_0^{s^2} \log_2^s q_0$, $\log_2 N \geq 2q_0^{s-1}C_1$. Then*

$$\inf_{\mathbf{x} \in [0,1]^s} \max_{1 \leq M \leq N} MD^*((H_{\mathcal{P}}^{\Sigma}(n, \mathbf{x}))_{n=1}^M) \geq C^{-1} \log_2^s N. \quad (2.1)$$

This result supports conjecture (1.3) (see also [Le1] and [Le3]).

First we will construct a double sequence $(\tau_{i,j})_{1 \leq i \leq s, j \geq 1}$. In order to construct $(\tau_{i,j})_{1 \leq i \leq s, j \leq 1}$, we define auxiliary sequences $\mathcal{L}_{i,j}^{(m)}$, $L_i^{(m)}$, $l_{i,j}$, $\mathcal{F}_{i,b}^{(m)}$, ... as follows.

2.1 Construction of the sequence $(\tau_{i,j})$.

Let $\mathbf{m} = [\log_{q_0}(N)/s - 1]$ with $q_0 = \max_{i,j} q_{i,j}$, $a_{i,j} \equiv \sigma_{i,j}^{-1}(0) - \sigma_{i,j}^{-1}(1) \pmod{p_{i,j}}$, $a_{i,j} \in \{1, \dots, p_{i,j} - 1\}$,

$$\mathcal{L}_{i,j,\tilde{a}_{i,j}}^{(m)} = \{1 \leq k \leq \mathbf{m} \mid p_{i,k} = q_{i,j}, \quad a_{i,k} = \tilde{a}_{i,j}\}, \quad (2.2)$$

$$L_i^{(m)} = \#\mathcal{L}_{i,g_{i,m},\tilde{a}_{i,m}}^{(m)} = \max_{1 \leq j \leq h_i, 1 \leq \tilde{a}_{i,j} < q_{i,j}} \#\mathcal{L}_{i,j,\tilde{a}_{i,j}}^{(m)}, \quad \text{where } g_{i,m} \in [1, h_i],$$

$\mathbf{a}_i = \mathbf{a}_{i,m} \in [1, q_{i,g_{i,m}} - 1]$, $1 \leq i \leq s$. We enumerate the set $\mathcal{L}_{i,g_{i,m},\mathbf{a}_i}^{(\mathbf{m})}$:

$$\mathcal{L}_{i,g_{i,m},\mathbf{a}_i}^{(\mathbf{m})} = \{l_{i,1} < \cdots < l_{i,L_i^{(\mathbf{m})}}\}.$$

We see that

$$L_i^{(\mathbf{m})} \geq \mathbf{m}/(h_i q_0) \quad \text{and} \quad a_{i,l_{i,j}} = \mathbf{a}_i, \quad i = 1, \dots, s, \quad j = 1, \dots, \mathbf{m}. \quad (2.3)$$

Let $p_i = p_i^{(\mathbf{m})} = q_{i,g_{i,m}}$, $p_0 = p_0^{(\mathbf{m})} = p_1 p_2 \cdots p_s$, $\dot{p}_i = p_0/p_i$ and

$$\mathcal{F}_{i,b}^{(\mathbf{m})} = \{1 \leq k \leq L_i^{(\mathbf{m})} \mid \tilde{P}_{i,l_{i,k}}^{-1} \equiv b \pmod{\dot{p}_i}\}. \quad (2.4)$$

We define F_i , m and $b_i = b_i^{(\mathbf{m})}$ as follows:

$$F_i = F_i^{(\mathbf{m})} = \#\mathcal{F}_{i,b_i}^{(\mathbf{m})} = \max_{0 \leq b < \dot{p}_i} \#\mathcal{F}_{i,b}^{(\mathbf{m})}, \quad m = \min_{1 \leq i \leq s} F_i^{(\mathbf{m})}. \quad (2.5)$$

It is easy to see that

$$m \geq \min_{1 \leq i \leq s} \mathbf{m}/(h_i q_0 \dot{p}_i) \geq \mathbf{m} h_0^{-1} q_0^{-s} \geq C_1^{-1} \log_2 N, \quad (2.6)$$

with $C_1 = 2s h_0 q_0^s \log_2 q_0$. We enumerate the set $F_{i,b_i}^{(\mathbf{m})}$:

$$\mathcal{F}_{i,b_i}^{(\mathbf{m})} = \{f_{i,1} < \cdots < f_{i,F_i}\}.$$

Let $\mathbf{k} = (k_1, \dots, k_s)$, $\tau_{i,j} = l_{i,f_{i,j}}$, $\boldsymbol{\tau}_{\mathbf{k}} = (\tau_{1,k_1}, \dots, \tau_{s,k_s})$, $P_{i,k} = \tilde{P}_{i,\tau_{i,k}}$,

$$P_{\mathbf{k}} = \prod_{i=1}^s P_{i,k_i}, \quad M_{i,\mathbf{k}} = \tilde{M}_{i,\boldsymbol{\tau}_{\mathbf{k}}}, \quad \text{with} \quad \tilde{M}_{i,\mathbf{k}} \equiv \prod_{1 \leq j \leq s, j \neq i} \tilde{P}_{j,k_j}^{-1} \pmod{\tilde{P}_{i,k_i}}. \quad (2.7)$$

By (2.4), we have that $(b_i, \dot{p}_i) = 1$ and $(b_j, p_i) = 1$ for $i \neq j$ ($i, j = 1, \dots, s$).

Let $c_i \equiv \prod_{1 \leq j \leq s, j \neq i} b_j \pmod{p_i}$. According to (2.3), (2.4) and (2.7), we obtain

$$(c_i, p_i) = 1, \quad M_{i,\mathbf{k}} \equiv c_i \pmod{p_i} \quad \text{and} \quad a_{i,\tau_{i,j}} = \mathbf{a}_i, \quad j \geq 1, \quad i = 1, \dots, s. \quad (2.8)$$

Let

$$\tilde{p}_i = g.c.d.(\mathbf{a}_i, p_i), \quad \hat{p}_i = p_i/\tilde{p}_i, \quad \hat{a}_i = \mathbf{a}_i/\tilde{p}_i, \quad d_i \equiv c_i \mathbf{a}_i \pmod{\hat{p}_i},$$

$d_i \in \{1, \dots, \hat{p}_i - 1\}$. Hence

$$d_i/\hat{p}_i \equiv c_i \mathbf{a}_i/p_i \pmod{1}, \quad (d_i, \hat{p}_i) = 1, \text{ and } \hat{p}_i > 1, \quad i = 1, \dots, s. \quad (2.9)$$

Let $\mathbf{m} = (m, \dots, m)$. From (1.5) and (2.7), we derive

$$2P_{\mathbf{m}} \leq 2 \prod_{i=1}^s \prod_{j=1}^{\tau_{i,m}} p_{i,j} \leq 2q_0^{\mathbf{m}s} \leq q_0^{s[s^{-1} \log_{q_0} N]} \leq N. \quad (2.10)$$

2.2 Using the Chinese Remainder Theorem.

Let $x_i = \sum_{j \geq 1} x_{i,j} \tilde{P}_{i,j}^{-1}$, with $x_{i,j} \in \{0, 1, \dots, p_{i,j} - 1\}$, $i = 1, \dots, s$. We define the truncation

$$[x_i]_r = \sum_{1 \leq j \leq r} x_{i,j} \tilde{P}_{i,j}^{-1} \quad \text{with } r \geq 1.$$

If $x = (x_1, \dots, x_s) \in [0, 1]^s$, then the truncation $[\mathbf{x}]_{\mathbf{r}}$ is defined coordinatewise, that is, $[\mathbf{x}]_{\mathbf{r}} = ([x_1]_{r_1}, \dots, [x_s]_{r_s})$, where $\mathbf{r} = (r_1, \dots, r_s)$.

By (1.6), we have

$$[\varphi_{\mathcal{P}_i}(n)]_{r_i} = [x_i]_{r_i} \Leftrightarrow n \equiv \sum_{1 \leq j \leq r} x_{i,j} \tilde{P}_{i,j-1} \pmod{\tilde{P}_{i,r}}.$$

Applying (2.7) and the Chinese Remainder Theorem, we get

$$[H_{\mathcal{P}}(n)]_{\mathbf{r}} = [\mathbf{x}]_{\mathbf{r}} \Leftrightarrow n \equiv \check{x}_{\mathbf{r}} \pmod{\tilde{P}_{\mathbf{r}}}, \quad (2.11)$$

$$\check{x}_{\mathbf{r}} \equiv \sum_{i=1}^s \tilde{M}_{i,\mathbf{r}} \tilde{P}_{\mathbf{r}} \tilde{P}_{i,r_i}^{-1} \sum_{1 \leq j \leq r} x_{i,j} \tilde{P}_{i,j-1} \pmod{\tilde{P}_{\mathbf{r}}}, \quad \check{x}_{\mathbf{r}} \in [0, \tilde{P}_{\mathbf{r}}). \quad (2.12)$$

It is easy to verify that if $r'_i \geq r_i$, $i = 1, \dots, s$, then

$$\check{x}_{\mathbf{r}'} \equiv \check{x}_{\mathbf{r}} \pmod{\tilde{P}_{\mathbf{r}}}. \quad (2.13)$$

According to (1.4), we get

$$\text{if } [\mathbf{w}]_{\mathbf{r}} = [\mathbf{x}]_{\mathbf{r}}, \quad \text{then } [T_{\mathcal{P}}^n(\mathbf{w})]_{\mathbf{r}} = [T_{\mathcal{P}}^n(\mathbf{x})]_{\mathbf{r}}, \quad n = 0, 1, \dots$$

From (1.4), (1.6) and (2.11), we obtain

$$[T_{\mathcal{P}}^W(\mathbf{0})]_{\mathbf{r}} = [H_{\mathcal{P}}(W)]_{\mathbf{r}} = [\mathbf{x}]_{\mathbf{r}}, \quad W = \check{x}_{\mathbf{r}}.$$

Hence

$$[T_{\mathcal{P}}^n(\mathbf{x})]_{\mathbf{r}} = [T_{\mathcal{P}}^n(T_{\mathcal{P}}^W(\mathbf{0}))]_{\mathbf{r}} = [T_{\mathcal{P}}^{n+W}(\mathbf{0})]_{\mathbf{r}} = [H_s(n+W)]_{\mathbf{r}}.$$

Let

$$W_{\mathbf{m}}(\mathbf{x}) := \check{x}_{\mathbf{m}} \in [0, P_{\mathbf{m}}]. \quad (2.14)$$

Therefore

$$[T_{\mathcal{P}}^n(\mathbf{x})]_{\mathbf{r}} = [H_{\mathcal{P}}(n + W_{\mathbf{m}}(\mathbf{x}))]_{\mathbf{r}} \quad 1 \leq r_i \leq m, \quad 1 \leq i \leq s, \quad n \geq 0. \quad (2.15)$$

2.3 Construction of boundary points y_1, \dots, y_s and u_1, \dots, u_s .

Let $\mathbf{y} = (y_1, \dots, y_s)$ with $y_i = \sum_{1 \leq j \leq m} P_{i,j}^{-1}$, and let $\ddot{y}_{i,k_i} = \sum_{1 \leq j \leq k_i} P_{i,j}^{-1}$, $k_i \geq 1$, $i = 1, \dots, s$, $\mathbf{k} = (k_1, \dots, k_s)$,

$$B_{\mathbf{y}} = [0, y_1) \times \dots \times [0, y_s), \quad B^{(\mathbf{k})} = \prod_{i=1}^s [\ddot{y}_{i,k_i} - P_{i,k_i}^{-1}, \ddot{y}_{i,k_i}). \quad (2.16)$$

We deduce

$$B_{\mathbf{y}} = \bigcup_{k_1, \dots, k_s=1}^m B^{(\mathbf{k})}, \quad \text{and } \mathbb{1}_{B_{\mathbf{y}}}(\mathbf{z}) - y_1 \cdots y_s = \sum_{k_1, \dots, k_s=1}^m (\mathbb{1}_{B^{(\mathbf{k})}}(\mathbf{z}) - P_{\mathbf{k}}^{-1}). \quad (2.17)$$

Let $\mathbf{u} = (u_1, \dots, u_s)$, $u_i = \sum_{j \geq 1}^{\tau_{i,m}} u_{i,j} \tilde{P}_{i,j}^{-1}$ with $u_{i,j} = \sigma_{i,j}^{-1}(y_{i,j})$, $u_{i,j}^* = \sigma_{i,j}^{-1}(0)$,

$$\mathbf{u}^{(\mathbf{k})} = (u_1^{(k_1)}, \dots, u_s^{(k_s)}) \quad \text{with} \quad u_i^{(k_i)} = \sum_{j=1}^{\tau_{i,k_i}-1} u_{i,j} \tilde{P}_{i,j}^{-1} + u_{i,\tau_{i,k_i}}^* \tilde{P}_{\tau_{i,k_i}}^{-1}, \quad (2.18)$$

$$\check{u}^{(\mathbf{k})} \equiv \sum_{i=1}^s M_{i,\mathbf{k}} P_{\mathbf{k}} P_{i,k_i}^{-1} \left(\sum_{j=1}^{\tau_{i,k_i}-1} u_{i,j} \tilde{P}_{i,j-1} + u_{i,\tau_{i,k_i}}^* \tilde{P}_{i,\tau_{i,k_i}-1} \right) \pmod{P_{\mathbf{k}}},$$

$$\check{u}_{\mathbf{k}} \equiv \sum_{i=1}^s M_{i,\mathbf{k}} P_{\mathbf{k}} P_{i,k_i}^{-1} \sum_{j=1}^{\tau_{i,k_i}} u_{i,j} \tilde{P}_{i,j-1} \pmod{P_{\mathbf{k}}}, \quad \check{u}^{(\mathbf{k})}, \check{u}_{\mathbf{k}} \in [0, P_{\mathbf{k}}).$$

According to (2.2)-(2.7), we have $p_{i,\tau_{i,k_i}} = p_i$, $k_i = 1, \dots, m$, $i = 1, \dots, s$.

By (2.2), we get $a_{i,\tau_{i,k_i}} \equiv \sigma_{i,\tau_{i,k_i}}^{-1}(0) - \sigma_{i,\tau_{i,k_i}}^{-1}(1) \equiv u_{i,\tau_{i,k_i}}^* - u_{i,\tau_{i,k_i}} \pmod{p_i}$.

From (2.8), we obtain $a_{i,\tau_{i,k_i}} = \mathbf{a}_i$, $k_i = 1, \dots, m$, $i = 1, \dots, s$. Hence

$$\check{u}^{(\mathbf{k})} \equiv \check{u}_{\mathbf{k}} + A_{\mathbf{k}} \pmod{P_{\mathbf{k}}}, \quad \text{where } A_{\mathbf{k}} \equiv \sum_{i=1}^s M_{i,\mathbf{k}} P_{\mathbf{k}} p_i^{-1} \mathbf{a}_i \pmod{P_{\mathbf{k}}} \quad (2.19)$$

with $A_{\mathbf{k}} \in [0, P_{\mathbf{k}})$.

Let $\mathbf{w} = (w_1, \dots, w_s) := H_{\mathcal{P}}^{\Sigma}(n, \mathbf{x}) = \tilde{\Sigma}(T_{\mathcal{P}}^n(\mathbf{x}))$.

We see from (2.16) and (2.18) that

$$\mathbf{w} \in B^{(\mathbf{k})} \Leftrightarrow w_{i,j} = y_{i,j}, \quad j \in [1, \tau_{i,k_i}), \quad w_{i,\tau_{i,k_i}} = 0, \quad i \in [1, s] \Leftrightarrow \sigma_{i,j}(w_{i,j}) = u_{i,j}$$

$$1 \leq j \leq \tau_{i,k_i} - 1, \quad \sigma_{i,j}(w_{i,\tau_{i,k_i}}) = u_{i,\tau_{i,k_i}}^*, \quad i = 1, \dots, s \Leftrightarrow [T_{\mathcal{P}}^n(\mathbf{x})]_{\tau_{\mathbf{k}}} = \mathbf{u}^{(\mathbf{k})}.$$

Applying (2.11), (2.12), (2.15), (2.18) and (2.19), we have

$$H_{\mathcal{P}}^{\Sigma}(n, \mathbf{x}) \in B^{(\mathbf{k})} \Leftrightarrow [T_{\mathcal{P}}^n(\mathbf{x})]_{\tau_{\mathbf{k}}} = \mathbf{u}^{(\mathbf{k})} \Leftrightarrow [H_{\mathcal{P}}(n + W_{\mathbf{m}}(\mathbf{x}))]_{\tau_{\mathbf{k}}} = \mathbf{u}^{(\mathbf{k})}$$

$$\Leftrightarrow n + W_{\mathbf{m}}(\mathbf{x}) \equiv \check{u}^{(\mathbf{k})} \pmod{P_{\mathbf{k}}} \Leftrightarrow n \equiv v_m + A_{\mathbf{k}} \pmod{P_{\mathbf{k}}},$$

where $v_m \equiv -W_{\mathbf{m}}(\mathbf{x}) + \check{\mathbf{u}}_{\mathbf{m}} \equiv -W_{\mathbf{m}}(\mathbf{x}) + \check{\mathbf{u}}_{\mathbf{k}} \pmod{P_{\mathbf{k}}}$ and $v_m \in [0, P_{\mathbf{m}})$.

Hence

$$H_{\mathcal{P}}^{\Sigma}(n, \mathbf{x}) \in B^{(\mathbf{k})} \Leftrightarrow n \equiv v_m + A_{\mathbf{k}} \pmod{P_{\mathbf{k}}}, \quad v_m \in [0, P_{\mathbf{m}}), \quad n \geq 0. \quad (2.20)$$

Completion of the proof of Theorem.

Lemma 1. *Let*

$$\alpha_m := \frac{1}{P_{\mathbf{m}}} \sum_{M=1}^{P_{\mathbf{m}}} \Delta(B_{\mathbf{y}}, (H_{\mathcal{P}}^{\Sigma}(n, \mathbf{x}))_{n=v_m}^{v_m+M-1}). \quad (2.21)$$

Then

$$\alpha_m = \sum_{1 \leq k_1, \dots, k_s \leq m} \left(\frac{1}{2} - \frac{A_{\mathbf{k}}}{P_{\mathbf{k}}} - \frac{1}{2P_{\mathbf{k}}} \right). \quad (2.22)$$

Proof. Let $\mathcal{H}_n := H_{\mathcal{P}}^{\Sigma}(n, \mathbf{x})$. Using (2.20), we have

$$\sum_{n=v_m+M_1P_{\mathbf{k}}}^{v_m+(M_1+1)P_{\mathbf{k}}-1} (\mathbb{1}_{B^{(\mathbf{k})}}(\mathcal{H}_n) - P_{\mathbf{k}}^{-1}) = 0 \quad (2.23)$$

and

$$\sum_{n=v_m+M_1P_{\mathbf{k}}}^{v_m+M_1P_{\mathbf{k}}+M_2-1} (\mathbb{1}_{B^{(\mathbf{k})}}(\mathcal{H}_n) - P_{\mathbf{k}}^{-1}) = \sum_{n \in [v_m, v_m+M_2)} (\mathbb{1}_{B^{(\mathbf{k})}}(\mathcal{H}_n) - P_{\mathbf{k}}^{-1})$$

$$= \sum_{n \in [v_m, v_m + M_2], n = v_m + A_{\mathbf{k}}} 1 - M_2 P_{\mathbf{k}}^{-1} = \mathbb{1}_{[0, M_2]}(A_{\mathbf{k}}) - M_2 P_{\mathbf{k}}^{-1},$$

with $M_1 \geq 0$ and $M_2 \in [0, P_{\mathbf{k}}]$, $M_1, M_2 \in \mathbb{Z}$.

From (1.1) and (2.17), we get

$$\begin{aligned} \Delta(B_{\mathbf{y}}, (\mathcal{H}_n)_{n=v_m}^{v_m+M-1}) &= \sum_{n=v_m}^{v_m+M-1} (\mathbb{1}_{B_{\mathbf{y}}}(\mathcal{H}_n) - y_1 \cdots y_s) \\ &= \sum_{k_1, \dots, k_s=1}^m \rho(\mathbf{k}, M), \quad \text{with} \quad \rho(\mathbf{k}, M) = \sum_{n=v_m}^{v_m+M-1} (\mathbb{1}_{B(\mathbf{k})}(\mathcal{H}_n) - P_{\mathbf{k}}^{-1}). \end{aligned} \quad (2.24)$$

By (2.21), we obtain

$$\alpha_m = \sum_{1 \leq k_1, \dots, k_s \leq m} \alpha_{m, \mathbf{k}}, \quad \text{with} \quad \alpha_{m, \mathbf{k}} = \frac{1}{P_{\mathbf{m}}} \sum_{M=1}^{P_{\mathbf{m}}} \rho(\mathbf{k}, M). \quad (2.25)$$

Bearing in mind (2.23)-(2.24), we derive

$$\begin{aligned} \alpha_{m, \mathbf{k}} &= \frac{1}{P_{\mathbf{m}}} \sum_{M_1=0}^{P_{\mathbf{m}}/P_{\mathbf{k}}-1} \sum_{M_2=1}^{P_{\mathbf{k}}} \left(\sum_{n=v_m}^{v_m+M_1 P_{\mathbf{k}}-1} (\mathbb{1}_{B(\mathbf{k})}(\mathcal{H}_n) - P_{\mathbf{k}}^{-1}) \right. \\ &\quad \left. + \sum_{n=v_m+M_1 P_{\mathbf{k}}}^{v_m+M_1 P_{\mathbf{k}}+M_2-1} (\mathbb{1}_{B(\mathbf{k})}(\mathcal{H}_n) - P_{\mathbf{k}}^{-1}) \right) = \frac{1}{P_{\mathbf{m}}} \sum_{M_1=0}^{P_{\mathbf{m}}/P_{\mathbf{k}}-1} \sum_{M_2=1}^{P_{\mathbf{k}}} \left(\mathbb{1}_{[0, M_2]}(A_{\mathbf{k}}) - M_2 P_{\mathbf{k}}^{-1} \right) \\ &= \frac{1}{P_{\mathbf{k}}} \sum_{M_2=1}^{P_{\mathbf{k}}} \left(\mathbb{1}_{[0, M_2]}(A_{\mathbf{k}}) - M_2 P_{\mathbf{k}}^{-1} \right) = \frac{P_{\mathbf{k}} - A_{\mathbf{k}}}{P_{\mathbf{k}}} - \frac{P_{\mathbf{k}}(P_{\mathbf{k}} + 1)}{2P_{\mathbf{k}}^2} = \frac{1}{2} - \frac{A_{\mathbf{k}}}{P_{\mathbf{k}}} - \frac{1}{2P_{\mathbf{k}}}. \end{aligned}$$

Using (2.25), we have

$$\alpha_m = \sum_{1 \leq k_1, \dots, k_s \leq m} \left(\frac{1}{2} - \frac{A_{\mathbf{k}}}{P_{\mathbf{k}}} - \frac{1}{2P_{\mathbf{k}}} \right).$$

Hence Lemma 1 is proved. \blacksquare

Lemma 2. *With notations as above,*

$$|\alpha_m| \geq \frac{m^s}{4p_0} \quad \text{for} \quad m \geq 2p_0. \quad (2.26)$$

Proof. From (2.8) and (2.19), we get

$$[0, 1) \ni \frac{A_{\mathbf{k}}}{P_{\mathbf{k}}} \equiv \sum_{1 \leq i \leq s} M_{i, \mathbf{k}} P_{\mathbf{k}} p_i^{-1} \mathbf{a}_i / P_{\mathbf{k}} \equiv \frac{c_1 \mathbf{a}_1}{p_1} + \cdots + \frac{c_s \mathbf{a}_s}{p_s} \pmod{1}.$$

Applying (2.9) and (2.22), we derive

$$\alpha_m = m^s \left(\frac{1}{2} - \{\alpha\} \right) - \sum_{1 \leq k_1, \dots, k_s \leq m} \frac{1}{2P_{\mathbf{k}}}, \quad \text{with} \quad \alpha = \frac{d_1}{\hat{p}_1} + \cdots + \frac{d_s}{\hat{p}_s}, \quad (2.27)$$

where $(d_i, \hat{p}_i) = 1$, $\hat{p}_i > 1$, $i = 1, \dots, s$. and $\{x\}$ is the fractional part of x . We have that if $\hat{p}_0 = \hat{p}_1 \hat{p}_2 \cdots \hat{p}_s \not\equiv 0 \pmod{2}$ then $\alpha \not\equiv 1/2 \pmod{1}$. Let $\hat{p}_\nu \equiv 0 \pmod{2}$ for some $\nu \in [1, s]$, and let $\alpha \equiv 1/2 \pmod{1}$. Then

$$(\hat{p}_\nu/2 - d_\nu)/p_\nu \equiv \sum_{1 \leq i \leq s, i \neq \nu} d_i/\hat{p}_i \pmod{1} \quad \text{and} \quad a_1 \equiv a_2 \pmod{p_0},$$

with $a_1 = \hat{p}_0(\hat{p}_\nu/2 - d_\nu)/\hat{p}_\nu$ and $a_2 = \sum_{i \neq \nu} \hat{p}_0 d_i/\hat{p}_i$. Let $j \in [1, s]$ and $j \neq \nu$. We see that $a_1 \equiv 0 \pmod{\hat{p}_j}$ and $a_2 \not\equiv 0 \pmod{\hat{p}_j}$. We get a contradiction. Hence $\alpha \not\equiv 1/2 \pmod{1}$. We have

$$\left| \frac{1}{2} - \{\alpha\} \right| = \left| \frac{1}{2} - \left\{ \left(\frac{d_1}{\hat{p}_1} + \cdots + \frac{d_s}{\hat{p}_s} \right) \right\} \right| = \frac{|a|}{2\hat{p}_0}, \quad \text{with some integer } a.$$

Thus $|1/2 - \{\alpha\}| \geq 1/(2\hat{p}_0) \geq 1/(2p_0)$ with $p_0 = p_1 \cdots p_s$, $(p_0, \hat{p}_0) = \hat{p}_0$.

Bearing in mind that $P_{\mathbf{k}} \geq 2^{k_1+k_2+\cdots+k_s}$, we obtain from (2.27) that

$$|\alpha_m| \geq \frac{m^s}{2p_0} - \frac{1}{2} = \frac{m^s}{2p_0} \left(1 - \frac{p_0}{m^s} \right) \geq \frac{m^s}{4p_0} \quad \text{for} \quad m \geq 2p_0. \quad (2.28)$$

Hence Lemma 2 is proved. \blacksquare

Going back to the proof of Theorem, by (2.1) and (2.6), we get

$$m^s (4p_0)^{-1} \geq (4p_0)^{-1} C_1^{-s} \log_2^s N = 2C^{-1} \log_2^s N, \quad \text{and} \quad m \geq C_1^{-1} \log_2 N \geq 2p_0,$$

where $C_1 = 2sh_0q_0^s \log_2 q_0$ and $C = (8p_0)^{-1} C_1^s = 2^{s+3} s^s h_0^s q_0^{s^2} \log_2^s q_0$.

Using (2.10) and (2.20), we have that $v_m + P_{\tau m} \leq 2P_{\mathbf{m}} \leq N$.

According to (2.28), (2.21) and (1.2), we obtain

$$2C^{-1} \log_2^s N \leq m^s (4p_0)^{-1} \leq |\alpha_m| \leq \sup_{1 \leq M \leq P_{\mathbf{m}}} MD^*((H_{\mathcal{P}}^{\Sigma}(n, \mathbf{x}))_{n=v_m}^{v_m+M-1})$$

$$\leq \sup_{1 \leq L, L+M \leq 2P_m} MD^*((H_{\mathcal{P}}^{\Sigma}(n, \mathbf{x}))_{n=L}^{L+M-1}) \leq 2 \sup_{1 \leq M \leq N} MD^*((H_{\mathcal{P}}^{\Sigma}(n, \mathbf{x}))_{n=1}^M).$$

Hence the Theorem is proved. ■

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